When Assumptions Make the Difference: The Curious Case of the Mistreated Bernoulli's Equation

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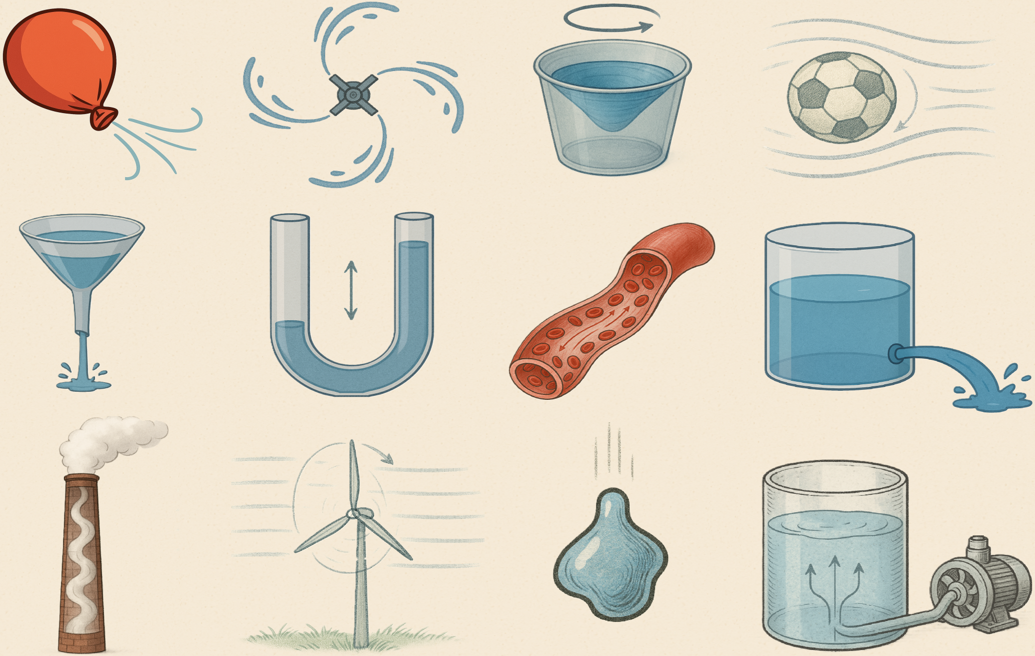
Which of the 12 configurations on the next slide allow the use of Bernoulli's equation?



A few considerations

- I was not precise enough. One must also specify: "**between which pair of points** can Bernoulli's equation be applied?" In fact, within the same system, there can be **regions** where Bernoulli's equation holds and others where it does not (see the third example)!
- **Bernoulli's Equation (BE): too many assumptions! Mistakes are just around the corner**.
- The confusion begins with the derivation itself:

High school derivation (energy conservation)	*University derivation (from NS Eqs.)*
There are no clear references to the underlying assumptions.	Assumptions are clearly invoked throughout the derivation.

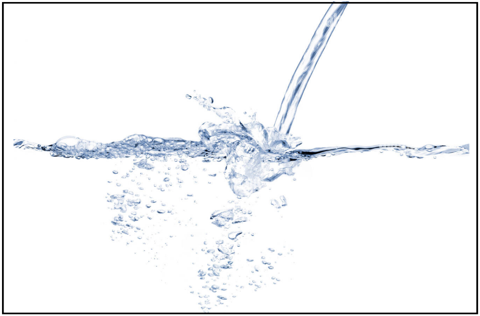
E. Marciotto, Phys. Educ. 51 045005 (2016)

Restricted Bernoulli's Equation

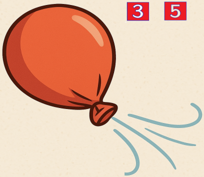
$$p(\vec{r}) + \frac{1}{2}\rho\, v^2(\vec{r}) + \rho\, \phi(\vec{r}) = \text{const.}$$

Working assumptions:

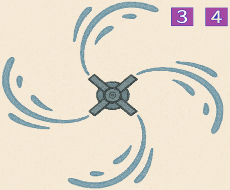
1. Inviscid fluid;
2. Incompressible fluid;
3. Steady flow;
4. Irrotational flow;
5. Laminar flow, i.e. smooth streamlines.
6. Fluid, i.e. ∞ collisions;
7. Inertial frame of reference;
8. Conservative external field.



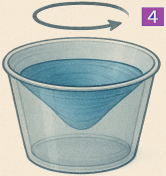

BADU-STREAM II



3 5



3 4



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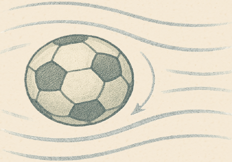
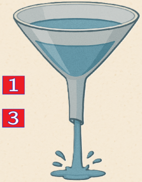
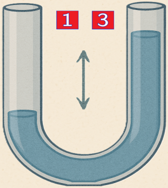


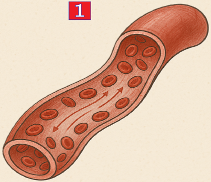
1 3 4 5



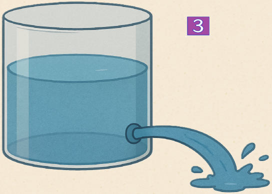
1 3



1 3

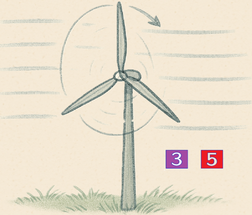


1



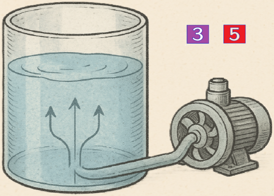
3

2



3 5

3



3 5

Extended Bernoulli's Equation

$$\int \left[\frac{\partial \vec{v}}{\partial t} + (\nabla \times \vec{v}) \times \vec{v}\right] \cdot \mathrm{d}\vec{s} + \int \left[\frac{\mathrm{d}^2\vec{\Lambda}}{\mathrm{d}t^2} + 2\vec{\Omega} \times \vec{v} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + \frac{\mathrm{d}\vec{\Omega}}{\mathrm{d}t} \times \vec{r}\right] \cdot \mathrm{d}\vec{s} +$$
$$+ \int \left[\frac{1}{\rho}\nabla p - \frac{1}{\rho}\nabla \cdot \overset{\leftrightarrow}{\tau}\right] \cdot \mathrm{d}\vec{s} + \frac{1}{2}v^2 + \phi = 0,$$

Working assumptions:

1. ~~Inviscid fluid~~;
2. ~~Incompressible fluid~~;
3. ~~Steady flow~~;
4. ~~Irrotational flow~~;
5. Laminar flow, i.e. smooth streamlines.

Extended Bernoulli's Equation

Very long expression$(\vec{r}) = $ const.

Working assumptions:

1. ~~Inviscid fluid~~;
2. ~~Incompressible fluid~~;
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4. ~~Irrotational flow~~;
5. Laminar flow, i.e. smooth streamlines.

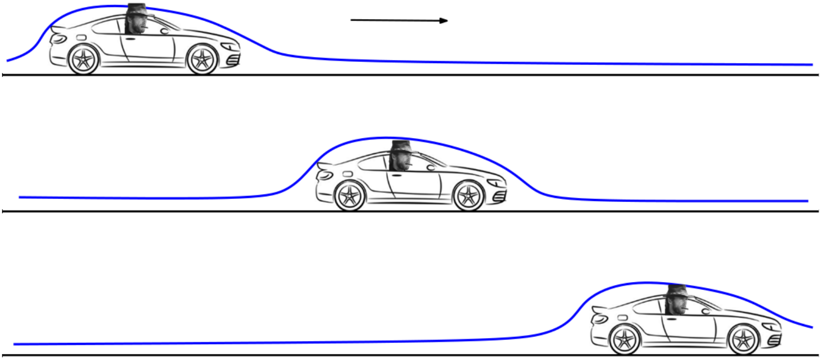
The smoker "paradox"

Suppose a smoker is driving in a car at constant velocity, with the windows slightly open. Does the smoke exit the car or stay inside?

- According to Blondie (the smoker), the air inside the car is at rest, while the outside air moves roughly at the car's velocity in the opposite direction. Blondie invokes the RBE: since higher speed implies lower pressure, the smoke exits the car.
- According to Tuco (standing on the ground), the air inside the car moves faster than the air outside. The RBE implies that the pressure outside is higher than inside, so the smoke remains inside the car.

So, who is right? Out of 50 Olympiad participants, none could answer!

The smoker "paradox": external observer viewpoint



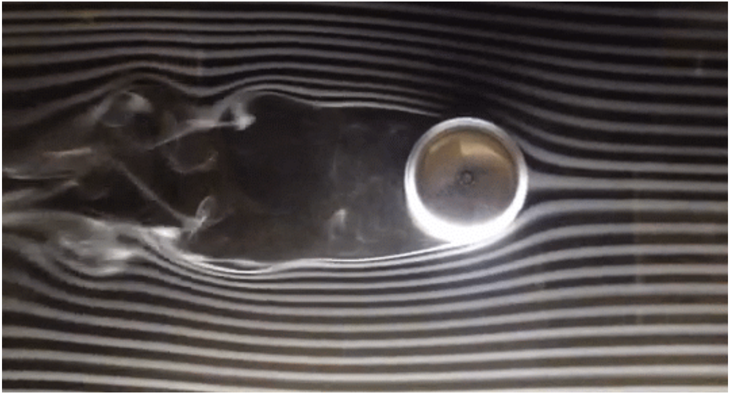
As seen by Tuco, the flow is unsteady!

The resolution of the "paradox"

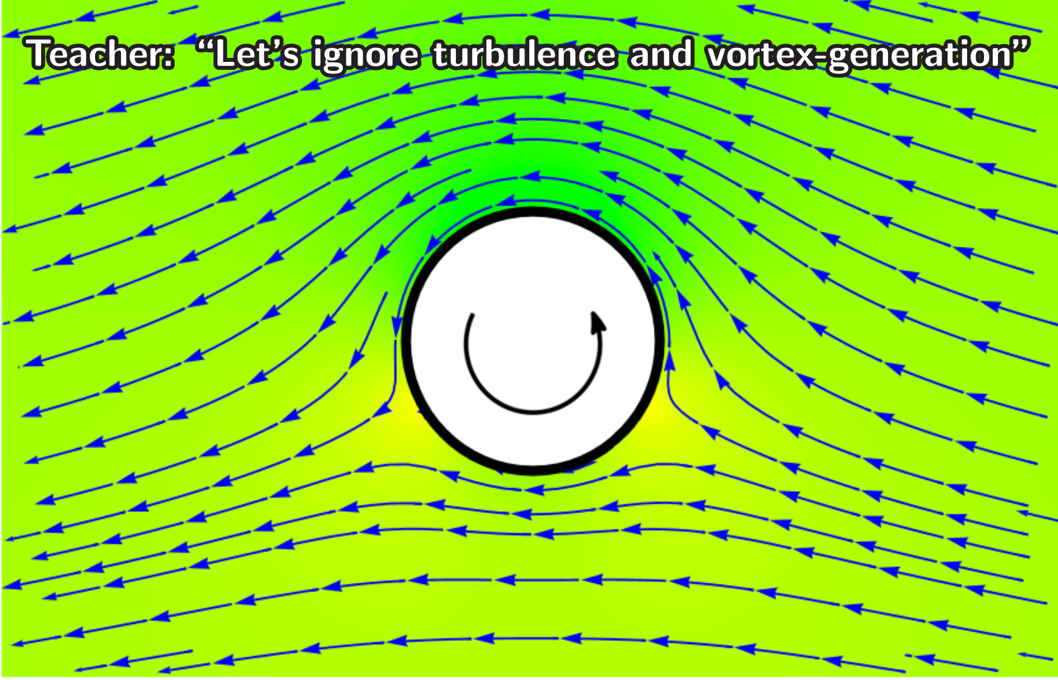
- Every observer not moving with the car sees an unsteady flow $\Rightarrow$ in their reference frame, the RBE cannot be applied!
- Blondie sees a flow that is always steady $\Rightarrow$ he can correctly apply the RBE $\Rightarrow$ the smoke exits the car!
- What happens if the car accelerates? There is no reference frame in which the flow is steady $\Rightarrow$ one must use the unsteady version of BE.
- **Take-home messages**:
  - Blondie is always right.
  - Smoking is bad.
  - Most paradoxes in fluid dynamics are due to a lack of attention to the assumptions. We are too lazy!

(Clarification: I referred to *smoke* in this example merely to make the discussion more engaging, but the correct question should actually be: *is the pressure inside the machine higher or lower than the atmospheric pressure?* In either case, air recirculation still occurs (and thus smoke would still leave the car). Moreover, turbulent and viscous effects have been neglected here.

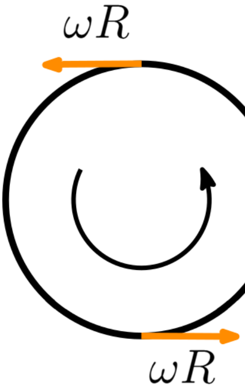
The great deceiver: Magnus effect



Unsteady, rotational and turbulent flow!


Teacher: “Let’s ignore turbulence and vortex-generation”

Magnus effect: the teacher's contradiction



1. Top:

$$p_0 + \frac{1}{2}\rho v^2 = p_{\text{top}} + \frac{1}{2}\rho(v+\omega R)^2$$

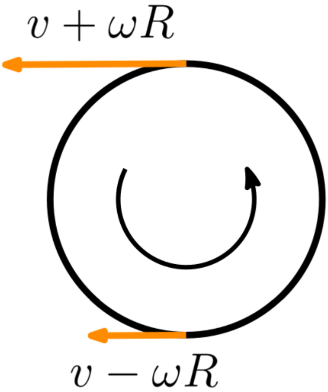
2. Bottom:

$$p_0 + \frac{1}{2}\rho v^2 = p_{\text{bot}} + \frac{1}{2}\rho(v-\omega R)^2$$

$$\Rightarrow p_{\text{bot}} - p_{\text{top}} = 2\rho v\omega R > 0.$$

$\Rightarrow$ The ball experiences an upward-directed force. However, we have assumed the existence of a boundary layer, which only occurs if the fluid is viscous. But if the fluid is viscous, the use of RBE is not permitted!

Magnus effect: the teacher's contradiction



1. Top:

$$p_0 + \frac{1}{2}\rho v^2 = p_{\text{top}} + \frac{1}{2}\rho(v + \omega R)^2$$

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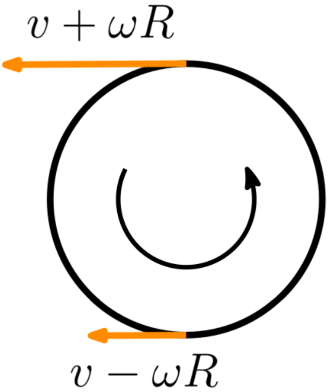

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Didactic value of the Magnus effect: Dimensional Analysis

- **Take-home message**: the explanation of the Magnus effect fails when one overlooks the complex effects due to vortex generation. RBE cannot explain it!
- Is everything lost? Not at all—this is actually a great opportunity to introduce dimensional analysis! The force acting on the ball is certainly independent of the mass of the ball, since it is a force exerted by the fluid through which the body moves.

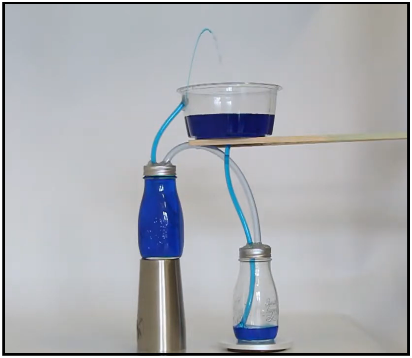
$$F = C\rho^{\alpha}\,\omega\,v^{\beta}R^{\gamma}.$$

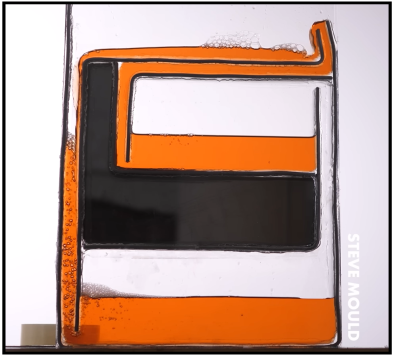
Switching to dimensions:

$$\frac{[L][M]}{[T]^2} = \left(\frac{[M]}{[L]^3}\right)^{\alpha}\frac{1}{[T]}\left(\frac{[L]}{[T]}\right)^{\beta}[L]^{\gamma},$$

which implies $\alpha = 1$, $\beta = 1$ and $\gamma = 3$, so that $F \propto \rho\,\omega\,v\,R^3$.

Heron's fountain and perpetual motion

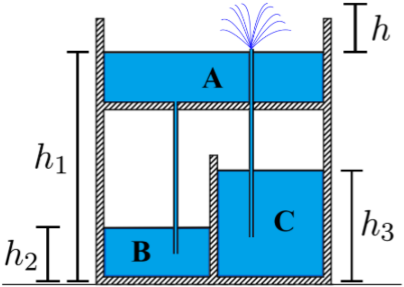




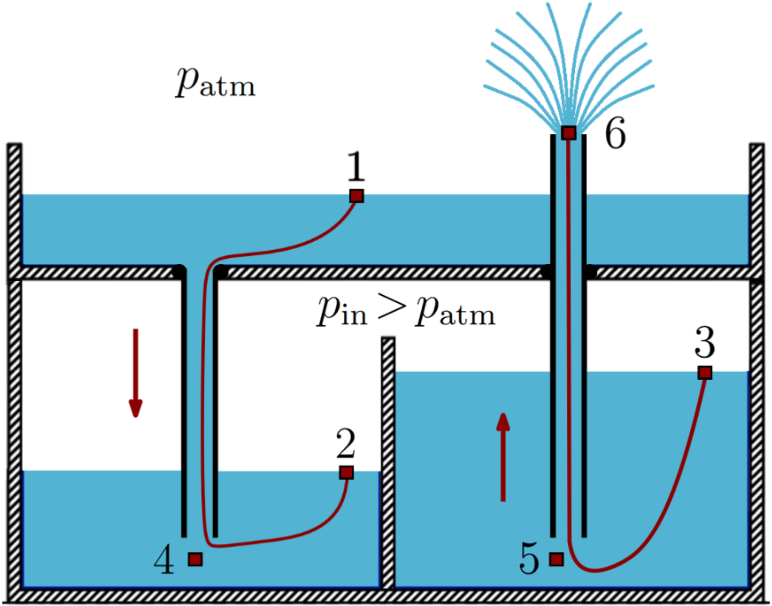
Physics Team Competition (GaS)

$\mathcal{P}7$ Heron's fountain

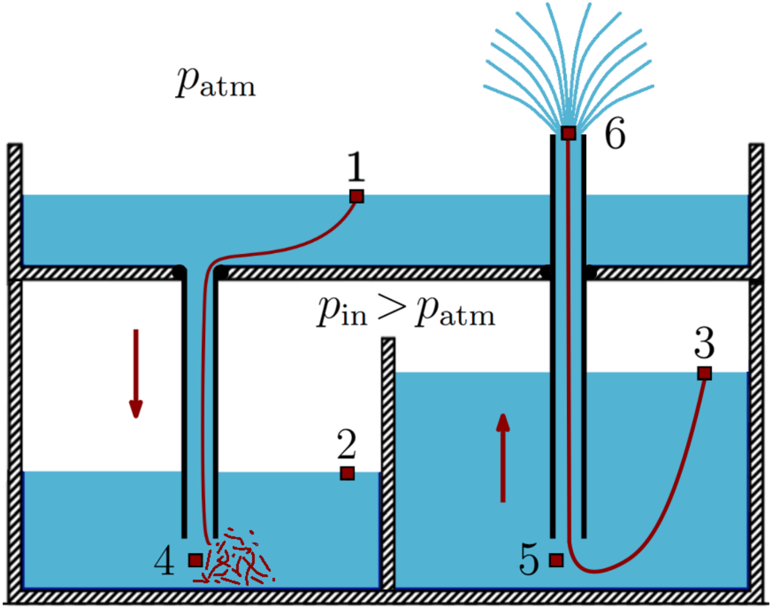
The figure on the right shows a schematic representation of *Heron's fountain*, a device invented by the famous Alexandrian engineer around the 2nd century AD. The upper chamber is open at the top and communicates with the lower chamber through two thin vertical pipes, whose cross-sections are negligible compared to those of the containers. The regions labeled A, B and C contain water, while D contains air at a pressure slightly higher than atmospheric pressure.



At a certain moment, the heights shown in the figure are $h_1 = 38.5\,\text{cm}$, $h_2 = 3.3\,\text{cm}$ and $h_3 = 12.7\,\text{cm}$, and the upward velocity in the vertical pipes is 7% lower than the downward velocity. Neglecting the effects of water viscosity, what is the height h of the water jet at that moment?



p_{atm}
1
6
$p_{\text{in}} > p_{\text{atm}}$
3
2
4
5



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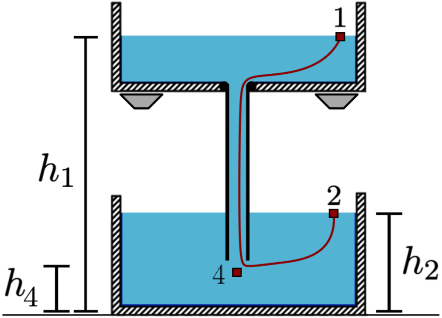

Wrong approach:

$$\cancel{p_1} + \cancel{\frac{1}{2}\rho v_1^2} + \rho g h_1 = \cancel{p_2} + \cancel{\frac{1}{2}\rho v_2^2} + \rho g h_2 \quad \Rightarrow \quad h_1 = h_2.$$

Correct approach:

$$p_1 + \cancel{\frac{1}{2}\rho v_1^2} + \rho g h_1 = [\, p_2 + \rho g (h_2 - h_4) \,] + \frac{1}{2}\rho v_4^2 + \rho g h_4 \quad \Rightarrow \quad v_4 = \sqrt{2g(h_1 - h_2)}.$$

Heron's fountain

Wrong equation (points $1 \to 2$)

$$p_{\text{atm}} + \cancel{\frac{1}{2}\rho v_1^2} + \rho g h_1 =$$
$$p_{\text{in}} + \cancel{\frac{1}{2}\rho v_2^2} + \rho g h_2$$

Energy is not conserved between points 4 and 2! **Submerged jet**: losses are due to turbulence, not to viscosity (Onsager's conjecture).

Correct equation (points $1 \to 4$)

$$p_{\text{atm}} + \cancel{\frac{1}{2}\rho v_1^2} + \rho g h_1 =$$
$$[p_{\text{in}} + \rho g(h_2 - h_4)] + \frac{1}{2}\rho v_4^2 + \rho g h_4$$

$$\Downarrow$$

$$v_4 \approx \sqrt{2g(h_1 - h_2) + 2(p_{\text{atm}} - p_{\text{in}})/\rho}$$

Pos	Squadra	Punti	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P11
			32	57	44	55	56	49	101	89	41	100	96
1	Lingangulguliguliwata	978	22	62	46	57	67	59	-10	110	42	242	-30
2	La Sagra Del Cacciucco	750	37	67	56	48	57			95	31	-20	105
3	Un Nome Qualunque	713	40	48	44		57	-10		-20	31	113	
4	Sagittario A	644	22	75	14	92	57	49		92	41		
5	Termodinianici	644	32		44	71	57	51	122	-10	41		106
6	Megaredi	617	32	49	34	-20	47	50		103	122		
7	Messe25 1	612	32	58	44	-10	74			-10	41	-20	
8	I Demoni Di Laplace	582	32	73	47	56	57	98		-20	41	-10	-10
9	Taramelli	567	32	68	90	57	60	69		-10	31	-10	
10	Gli Elettricisti	556	22		37	56		53		142	41		
11	Donegani 1	554	44	58	100	56	57	49		-20	47	-30	
12	Galilei 1	549	32		-10	59				71	41	236	-10
13	S'Ei Vince, Ei Lice	526	33	48	49	49	57	78		-30	41		
14	Le Mele Di Newton	522	32	64	47		57	49			41		
15	Galiscopici	510	12	58	57	46	72		-10	-30	41	-20	
16	Gb Grassi 1	504	33	58	44	36		128		-10	44	-20	
17	Matematico Chi Perde!	503	22		-10		50	41			56		224
18	Voltaminchioni	497	34	58	34	104		52		-20	31	104	-30
19	Calini A	493	32	58	54		37	56		-10	21	-20	-10
20	Paleo_35_Sa	492	52	58	48	64	-10	49			41		
21	Respo Team 1	479	32		45	76				-20	102		114
22	Squadra 4	478	32	48	44	94	57	49		-30	54		
23	Corpo Nero	477	47	38	-10	56	-30	49			90		107
24	Piccionebuco	475	22		44		57	50		91	41	-40	
25	Gli Amici Di Gauss	465	64	51						83	41		
26	Abs	458		58	68	65	47	49			41		
27	Gli Altri	457	32		4	56		49			116		
28	Risikati	451	64	58	44	56	58				41		
29	La Mentafisica	451	49	70	34	46	57	52		-20	53		-20
30	Squadra 5	448	32	132	24	46	-10			-10	41		-10
31	Le Due Torri	446	32	-10		36	47				41		
32	Parentucelli 1	446	32		34	118	57	49		-10	46	-10	

Pos	Squadra	Punti	P1 32	P2 57	P3 44	P4 55	P5 56	P6 49	P7 101	P8 89	P9 41	P10 100	P11 96
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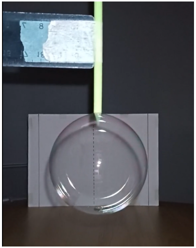
27	Gli Altri	457	32		4	56		49			116		
28	Risikati	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		[illegible]	[illegible]	[illegible]	[illegible]
29	La Me[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		[illegible]	[illegible]	[illegible]	-20
30	Squadra 5	448	32	132	24	46	-10			-10	41		-10
31	Le Due Torri	446	32	-10		36	47				41		
32	Parentucelli 1	446	32		34	118	57	49		-10	46	-10	

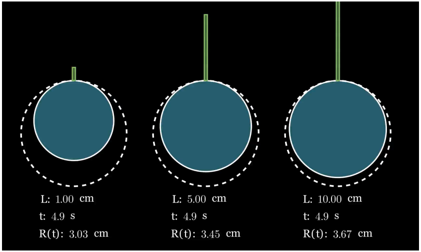
Just ten attempts and one correct answer!

How to improve?

The key lies in making pupils marvel through experiments! Fluid dynamics is the branch of physics that, more than any other, offers the most fascinating experimental breakthroughs. This year, the experimental examination of the Italian Physics Olympiad consisted on deflating soap bubbles!

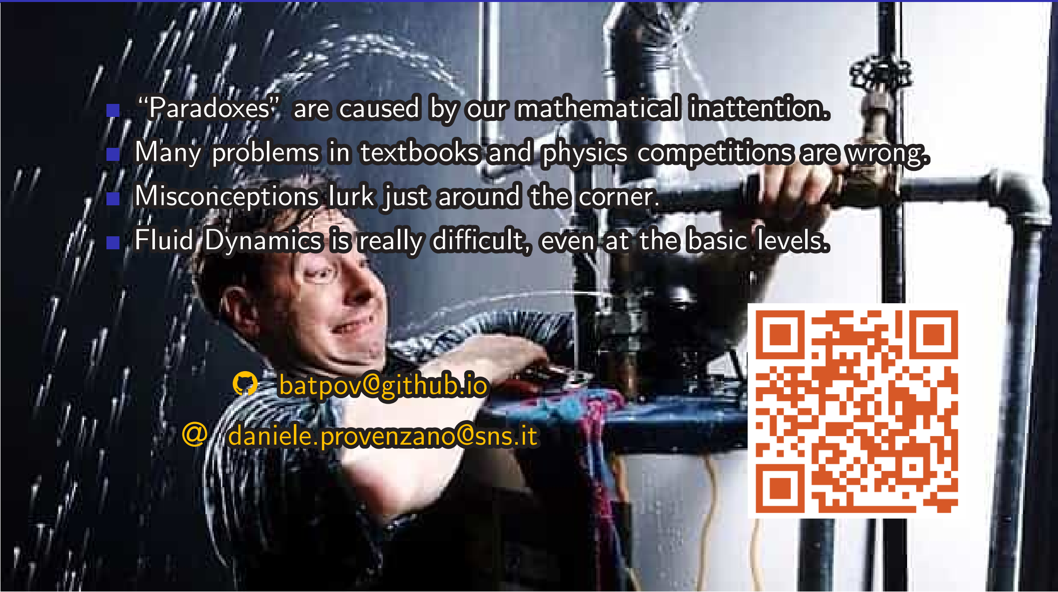


Experiment by A. Stefanini



Simulation by A. Cilione

Final thoughts



- "Paradoxes" are caused by our mathematical inattention.
- Many problems in textbooks and physics competitions are wrong.
- Misconceptions lurk just around the corner.
- Fluid Dynamics is really difficult, even at the basic levels.

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